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New BPS Configurations of BMN Matrix Theory

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Abstract: We explore the 1/2 BPS configurations in BMN matrix theory with $SO(3)$ angular momentum of $SO(3) \times SO(6)$ symmetry. The fluctuation analysis of the BPS configurations near the abelian solutions and also the fuzzy two sphere vacua reveals how nonabelian BPS configurations emerge. Especially the irreducible nonabelian configurations seem to have the maximal angular momentum of order N^3, beyond which they collapse to abelian ones. We also find some new BPS configurations explicitly.

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1. Introduction

BMN matrix theory with $U(N)$ gauge group has been proposed as the DLCQ limit of M theory on the 11-dim plane wave background with the maximal supersymmetry [1]. The action of this model can be also obtained from the matrix regularization of the membrane action in the pp-wave background, or the quantum mechanics of D0 branes on the background of the 11-d pp-wave compactified to 10-dim [2], or by dimensional reduction of 4-dim susy Yang-Mills theory on $R \times S^3$ [3]. Each vacuum of the BMN matrix theory is characterized by a partition of N and describes concentric fuzzy 2-spheres, which can be interpreted as giant gravitons in the pp-wave background.

The matrix theory has $SO(3) \times SO(6)$ symmetry and these fuzzy 2-sphere vacua can have also $SO(3)$ angular momentum, breaking the supersymmetry partially. While the BPS configurations with $SO(6)$ angular momentum are trivial ones, the BPS configurations with $SO(3)$ angular momentum are highly nontrivial. Only a few exact BPS configurations have been found for finite N [4, 5, 6]. In the infinite N (continuum) limit the general BPS configurations have been found to be Riemann surface with arbitrary number of genus and spikes [6]. The BPS configurations for finite N, which are expected to be a special class of fuzzy Riemann surfaces, have not been understood in general. (see however Ref. [7].)

In this work, we investigate these 1/2 BPS configurations carrying $SO(3)$ angular momentum for finite N in BMN matrix theory. A detailed fluctuation analysis of the BPS configurations near the abelian BPS configurations and also near the nonabelian ground states leads to new insight on how nonabelian BPS configurations emerge. Based on this analysis, we make several observations about these 1/2 BPS configurations of the BMN matrix theory. One is about the maximum value of the

$SO(3)$ angular momentum for any irreducible nonabelian configuration, which is defined as one which cannot be expressed as a sum of commuting configurations). We also find a new explicit class of BPS configurations for higher N.

The simplest BPS configurations with $SO(3)$ angular momentum are abelian solutions which are present even in $U(1)$ theory. In the $U(N)$ theory, the abelian BPS configuration would be made out of the time-dependent field configurations which are all simultaneously diagonaliable. The next simplest ones are the ellipsoidal solutions which are time-dependent configurations built on vacuum fuzzy spheres [4]. These fuzzy ellipsoidal solutions become abelian when the angular momentum, say J_3, exceeds a critical value of order N^3. (A similar phenomena concerning the maximal angular momentum of nonabelian angular momentum has been observed recently, in a somewhat different context [8].)

For small finite N, some toroidal BPS configurations have previously been found explicitly [5, 6]. All 1/2 BPS solutions with finite $SO(3)$ angular momentum have been also found in the continuum limit with infinite N [6]. These continuum BPS configurations consist of all possible genus surfaces maybe with some spikes. Our BPS equations do not seem integrable for finite N. All BPS solutions for $N = 2$ are known but we think that not all the solutions with $N = 3$ are known. We do a detailled analysis of the BPS configurations lying near abelian solutions or vacuum fuzzy spheres. This analysis shows that nonabelian BPS configurations emerge from the known solutions in a very specific way.

BMN matrix theory and BPS states have been analyzed in detail from the superalgebra analysis and the perturbative approach of the BPS states by expanding the theory around each vacuum in the large μ limit [2, 10, 11]. The infinite μ limit is a free theory and the interaction is of order $1/\mu$. This consideration does not directly involve the classical BPS configurations, which are nonperturbative. The protected spectrum of the matrix theory in the large N limit is also related to the linear fluctuation spectrum of the spherical M5-brane [12]. Our BPS configurations in the large N limit should also play a role in this context.

There are several degenerate vacua in the theory which are separated by an energy barrier and the tunnelling between these vacua has been studied in detail [13]. As there are enough supersymmetries, there is no lift of the vacuum energy due to the mixing by tunnelling. Similar tunnelling would also manifest itself for the BPS configurations for a given charge as there exist a lot of BPS configurations which are separated by energy barrier. The quantization of the 1/2 BPS configurations taking into account the tunnelling effect would be interesting. Still, we expect many quantum BPS states for a given N and the BPS angular momentum J_3. This leads to a natural index for counting the BPS states as a function of N and J_3.

In the context of Yang-Mills theory on $R \times S^3$, the 1/2 BPS states with $SO(3)$ angular momentum would correspond to the BPS chiral conformal operators with spherical angular momentum. As all of fuzzy sphere vacua in the matrix theory

are gauge equivalent to the trivial vacuum in the Yang-Mills theory, our 1/2 BPS solution would correspond to a special class of the 1/2 BPS solutions which has spatial angular momentum J_3. Most of the study of BMN matrix theory in the Yang-Mills theory context focuses on the cases of $SO(6)$ angular momentum, and it would be nice to extend this analysis to our case. (See for example Ref. [14] for a review.)

The plan of this work is as follows. In Sec.2, we review some basic aspects of BMN matrix theory and introduce irreducible BPS configurations. In Sec.3, we perform a perturbative analysis of the BPS configurations near the abelian solutions and the fuzzy sphere vacua. In Sec.4, we find some new solutions by writing the known solution in higher dimensional representations as well as find some genuinely new solutions. In Sec.5 we conclude with some remarks.

2. Plane-Wave Matrix Theory

In this work, we consider a special class of 1/2 BPS configurations in BMN matrix theory. BMN matrix theory has $SO(3) \times SO(6)$ global symmetries and we are here interested the classical BPS configuration with only $SO(3)$ angular momentum. Thus we focus on the part of BMN matrix gauge theory for three $N \times N$ hermitian matrices $X_a, a = 1, 2, 3$, whose Lagrangian is

$$L = \frac{1}{2}\mathrm{Tr}(D_0 X_a)^2 - U(X)\,, \tag{2.1}$$

where $D_0 X_a = \dot{X}_a - i[A_0, X_a]$, and the potential is

$$U(X) = \frac{1}{2}\,\mathrm{Tr}\left(\frac{\mu}{3}X_a + \frac{i}{2}\epsilon_{abc}[X_b, X_c]\right)^2 \tag{2.2}$$

with $\mu > 0$ by a choice of convention. (Here we have scaled out the other parameters in the theory for simplicity.) There is a local time-dependent $U(N)$ gauge symmetry, and any physical configuration must satisfy the Gauss law constraint,

$$\sum_a [X_a, D_0 X_a] = 0\,. \tag{2.3}$$

The conserved energy is

$$H = \frac{1}{2}\mathrm{Tr}(D_0 X_a)^2 + U(X)\,. \tag{2.4}$$

There is a SO(3) global rotation symmetry of three matrices X_1, X_2, X_3, whose conserved quantities are

$$J_a = \epsilon_{abc}\mathrm{Tr} X_b D_0 X_c\,. \tag{2.5}$$

The vacuum configurations are the minima of the potential $U(X) = 0$ and satisfy the equation

$$[X_a, X_b] = \frac{i\mu}{3}\epsilon_{abc}X_c\,. \tag{2.6}$$

With the scaling $X_a = \frac{\mu}{3}L_a$, L_a form the $SU(2)$ algebra. A partition $(p_1, p_2, , , , , p_K)$ of the number N with natural numbers p_k such that $\sum_k p_k = N$ characterizes a unique gauge equivalent classical vacuum where each p_k denotes the dimension of the irreducible representation of the $SU(2)$ generators L_a. For example for $N = 3$, we have the symmetric vacuum $(1, 1, 1)$ where L_a becomes three trivial 1-dim representations, the $(2, 1)$ vacuum where L_a forms one 2-dim irreducible representation and one trivial 1-dim one, and the (3) vacuum where L_a is the 3-dim irreducible representation. For general N, the symmetric phase $L_a = 0$ would be denoted as the $(1, 1, ..., 1)$ vacuum , and the maximally broken vacuum, where L_a is the N-dim irreducible representation, would be denoted as the (N) vacuum. The number of partitions would be the number of the gauge-equivalent vacua. As N increases, the number of the partition of N grows very fast and so is the number of gauge inequivalent vacua.

For a given conserved value, say, J_3, the energy can be reexpressed as

$$\begin{aligned} H = \frac{1}{2}\Big(D_0X_1 \pm (\frac{\mu}{3}X_2 + i[X_3, X_1])\Big)^2 + \Big(D_0X_2 \mp (\frac{\mu}{3}X_1 + i[X_2, X_3])\Big)^2 \\ +\frac{1}{2}(D_0X_3)^2 + \frac{1}{2}\Big(\frac{\mu}{3}X_3 + i[X_1, X_2]\Big)^2 \ \pm\frac{\mu}{3}J_3\,. \end{aligned} \tag{2.7}$$

Thus there is a BPS bound on the energy

$$H \geq \frac{\mu}{3}|J_3|\,, \tag{2.8}$$

where the Noether charge J_3 plays the role of a central term. (The charge J_3 is really the so-called 'non-central' term as the supercharge is not invariant under it [1].) The bound is saturated for the so-called BPS configurations which should satisfy the Gauss law constraint (2.3) and the following BPS equations;

$$\begin{aligned} &D_0X_1 = \pm\Big(-\frac{\mu}{3}X_2 - i[X_3, X_1]\Big)\,, \quad D_0X_2 = \pm\Big(\frac{\mu}{3}X_1 + i[X_2, X_3]\Big)\,, \\ &D_0X_3 = 0, \quad \frac{\mu}{3}X_3 + i[X_1, X_2] = 0\,. \end{aligned} \tag{2.9}$$

The upper sign is for $J_3 > 0$ and the lower sign is for $J_3 < 0$. For BPS configurations, the central charge becomes

$$J_3 = \pm\frac{\mu}{3}\ \mathrm{Tr}(X_1^2 + X_2^2 - 2X_3^2)\,. \tag{2.10}$$

Thus a BPS configuration of finite energy should satisfy the inequality

$$\mathrm{Tr}(X_1^2 + X_2^2) \geq 2\mathrm{Tr}(X_3^2)\,. \tag{2.11}$$

As J_3 is related to the rotation in the X_1, X_2 plane, the above inequality is consistent with the notion that that the BPS solutions are stretched along the 12 plane due to the centrifugal force.

To find nontrivial solutions, we start with the gauge choice $A_0 = X_3$ in Eq. (2.9), and read off the time dependence of X_a. We introduce a complex matrix W and a hermitian matrix Z, both of which are time-independent, and rewrite X_a as follows;

$$X_1 + iX_2 = \frac{\mu}{3} e^{\frac{i\mu t}{3}} W \,, \quad X_1 - iX_2 = \frac{\mu}{3} e^{-\frac{i\mu t}{3}} \bar{W} \,, \quad X_3 = \frac{\mu}{3} Z \,. \tag{2.12}$$

The BPS equations and Gauss law in Eqs.(2.9) and(2.3) become $[W, \bar{W}] = 2Z$ and

$$[W, [\bar{W}, Z]] + [\bar{W}, [W, Z]] = 4Z \,. \tag{2.13}$$

These equations are invariant under $SU(N)$ gauge transformations. In addition, there is an overall $U(1)$ phase rotation of W generated by the charge J_3. A simple solution of the above equations is

$$W = L_1 + iL_2 = L_+ \,, \bar{W} = L_1 - iL_2 = L_- \,, \quad [W, \bar{W}] = 2L_3 \,. \tag{2.14}$$

This solution describes the vacuum solution as the time-dependent X_1, X_2, X_3 in Eq.(2.12) are gauge transforms of the vacuum solution $X_a = \mu L_a/3$ by an unitary transformation e^{itL_3}. The vacuum solution has zero central charge.

We are interested in BPS configurations with nonzero charge J_3. There are many vacua, all of which are separated by some potential energy barrier. Let us imagine to add a bit of J_3 charge at a given vacuum and find the corresponding BPS configurations in a given vacuum. We expect many gauge inequivalent BPS configurations built in a given vacuum for a given charge J_3. Some of these solutions may be continuously connected to each other for a given charge. As we increase the charge, the solutions disconnected from each other may be get connected as, for example, the configurations goes over the energy barrier.

To study these BPS configurations, let us start from the abelian vacuum $(1, 1, 1, ..., 1)$ where $X_a = 0$, and add some J_3 charge. One can easily find that the purely abelian configurations where W is diagonal and traceless satisfy the BPS equations (2.13) with $Z = 0$. We parameterize the abelian solution as

$$W = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_N) \,. \tag{2.15}$$

The configuration is completely lying in the 12 plane as $X_3 = 0$. The central charge becomes

$$J_3 = \frac{\mu^3}{27} \sum_k |\lambda_k|^2 \,. \tag{2.16}$$

There is no upper bound on the value of J_3. For a given J_3, the configuration space is parameterized by N complex numbers. We will do a small fluctuation analysis

of the above abelian solutions in the next section to find out whether they can be deformed to nonabelian configurations.

To find nontrivial solutions, let us start from a vacuum $(p_1, p_2, .., p_K)$ and add a small J_3 charge. There may be several possible BPS solutions. The simplest one is the $SU(2)$ type where X_a is a linear combination of L_a whose representation is characterized by the partition $(p_1, p_2, ..., p_K)$. We start with W being a linear combination of L_a. By a $SU(2)$ gauge transformation, we choose $[W, \bar{W}]$ to be proportional to L_3, which is diagonal. This in turn implies that W should be a linear combination of L_1, L_2. Then the solution of the BPS equation becomes [4]

$$W = c_1 L_+ + c_2 L_-, \quad \bar{W} = \bar{c}_1 L_- + \bar{c}_2 L_+, \quad Z = (2|c_1|^2 - 1) L_3 \,, \tag{2.17}$$

where $|c_1|^2 + |c_2|^2 = 1$. Now we can make a $e^{i\alpha L_3}$ gauge rotation and a $W \to e^{i\beta} W$ spatial rotation to make both a, b real. Then the solution becomes the well-known ellipsoid solution

$$\frac{X^2}{(c_1 + c_2)^2} + \frac{Y^2}{(c_1 - c_2)^2} + \frac{Z^2}{(c_1^2 - c_2^2)^2} = \sum_c (L_c)^2 \,. \tag{2.18}$$

Since $c_1^2 + c_2^2 = 1$, the conserved charge J_3 becomes

$$J_3 = \frac{\mu^3}{27} \cdot \frac{8}{3} c_1^2 (1 - c_1^2) \sum_a \mathrm{Tr}(L_a)^2 \,. \tag{2.19}$$

Here we have used that $\mathrm{Tr} L_1^2 = \mathrm{Tr} L_2^2 = \mathrm{Tr} L_3^2 = \sum_a \mathrm{Tr}(L_a)^2/3$. Note that the case where $c_1 = 1, c_2 = 0$ or $c_1 = 0, c_2 = 1$ is the vacuum solution and the case where $c_1^2 = c_2^2 = 1/2$ is the configuration of the collapsed ellipsoid to an abelian thin line which is rotating on 12 plane. The central charge J_3 takes its maximal value at this abelian limit. Thus this solution is connected to the abelian solution at the maximal J_3 limit.

For the N-dim irreducible representation, $\mathrm{Tr} L_c^2 = N(N^2 - 1)/4$. For other representations $(p_1, p_2, ..., p_K)$ such that $\sum_k p_k = N$, $\mathrm{Tr} L_c^2 = \sum_k p_k (p_k^2 - 1)/4 \leq N(N^2 - 1)/4$. Thus, among the $SU(2)$ type solutions from the various vacua, the one with with N-dim irreducible representation from the (N) vacuum takes the maximal value,

$$J_{max} = \frac{\mu^3}{27} \frac{N(N^2 - 1)}{6} \,, \tag{2.20}$$

at the abelian limit.

One can easily generalize the above two types of solutions by putting them together. For any two commuting solutions $W = W_1, W_2$ of the BPS equations so that $[W_1, W_2] = 0, [W_1, \bar{W}_2] = 0$, their sum $W = W_1 + W_2$ is also a BPS configuration. Thus in the vacuum where L_a is not irreducible, we can generalize the above ellipsoid solution so that each irreducible part has a different parameter c_1 and also one can

add abelian solutions which commute with this generalized nonabelian solution. This leads to a division of all BPS configurations into reducible ones and irreducible ones. The irreducible ones are those which cannot be expressed as a sum of commuting BPS solutions. Thus all abelian solutions with $N \geq 2$ are reducible.

As an example for a reducible BPS configuration, we can consider a mixed-type BPS configuration built from the $(2, 1)$ vacuum in the $N = 3$ case, which is

$$W = \begin{pmatrix} c_3 & c_1 & 0 \\ c_2 & c_3 & 0 \\ 0 & 0 & -2c_3 \end{pmatrix} , \tag{2.21}$$

where $c_1^2 + c_2^2 = 1$ and c_3 is arbitrary. Then $Z = (c_1^2 - 1/2)\ \mathrm{diag}(1, -1, 0)$. When $c_1^2 = c_2^2 = 1/2$, the above solution can be diagonalized. When $c_1 = 1, c_2 = c_3 = 0$, it becomes the vacuum $(2, 1)$. Its angular momentum is

$$J_3 = \frac{\mu^3}{27}(6c_3^2 + 4c_1^2(1 - c_1^2)) . \tag{2.22}$$

Note that c_3 can be arbitrary and so there is no bound on J_3 for this solution. One can generalize this type of solution easily. For any nonabelian vacuum where L_a is not irreducible, there is at least one unbroken abelian $U(1)$ subgroup which commutes with L_a, and so one can add the J_3 charge into both abelian and nonabelian sectors. These type of solutions are reducible and can be decomposed to a sum of irreducible ones.

We are interested in all BPS configurations. Besides the ellipsoidal solutions, one may wonder whether there are other type of nonabelian solutions built on a given nonabelian vacuum by adding small angular momentum. In the next two sections, we will add some new understanding on this topic.

3. Fluctuation Analysis

For example, the ellipsoidal solution (2.17) collapses to the abelian solution at its maximal J_3 value. Thus one suspects that all nonabelian solutions are connected to abelian solutions (2.15). To find out this connection, let us make a small perturbation of the abelian solution,

$$W = \lambda_i \delta_{ij} + \epsilon_{ij}, \quad \bar{W} = \lambda_i^* \delta_{ij} + \epsilon_{ji}^* , \tag{3.1}$$

where $\epsilon_{ii} = 0$ for each i. Note that a pure gauge transformation would be $\epsilon_{ij} = (\lambda_i - \lambda_j)\chi_{ij}$ with antihermitian χ_{ij}. Then

$$[W, \bar{W}]_{ij} = (\lambda_i - \lambda_j)\epsilon_{ji}^* - (\lambda_i^* - \lambda_j^*)\epsilon_{ij} . \tag{3.2}$$

Then the left-hand-side of the BPS equation (2.13) becomes

$$l.h.s. = 2|\lambda_i - \lambda_j|^2\Big((\lambda_i - \lambda_j)\epsilon^*_{ji} - (\lambda^*_i - \lambda^*_j)\epsilon_{ij}\Big)\,, \tag{3.3}$$

which should be $4[W, \bar{W}]$. Thus the BPS equation is satisfied if for all non-vanishing ϵ_{ij} which is not a pure gauge transformation,

$$|\lambda_i - \lambda_j|^2 = 2\,. \tag{3.4}$$

The central charge does not change from the abelian result to the first order in perturbation. This analysis shows how any nonabelian configuration may be connected to the abelian solutions. This analysis does not tell whether an irreducible nonabelian solution becomes abelian as J_3 increases or decreases at some critical value, nor tell whether an irreducible nonabelian solution becomes abelian at once, or piece-wise.

Let us now consider how the above analysis appears in the the ellipsoid case (2.17). With N-dim irreducible L_a, the solution becomes abelian when $c_1 = c_2 = 1/\sqrt{2}$ as $W = \sqrt{2}L_1$. The solution $W = \sqrt{2}L_1$ can be diagonalized to be

$$W = \sqrt{2}\mathrm{diag}(l, l-1, l-2, ..., -l+1, -l)\,, \tag{3.5}$$

where $2l + 1 = N$. This solution satisfies the criterion (3.4). The ellipsoid solution (2.17) near this abelian solution becomes $W = (1-\epsilon^2)L_1 + \epsilon L_2$ which has nonzero $\epsilon_{i,i+1}$.

There are other possible cases. For example, for a given N, we could put λ_i on a circle whose center is at the origin. We require $|\lambda_n - \lambda_{n+k}|^2 = 2$ for a fixed k and $\lambda_{N+n} = \lambda_n$. We use the criteria (3.4) to get

$$\lambda_n = \frac{1}{\sqrt{2}\sin\frac{\pi k}{N}} e^{\frac{2i\pi nk}{N}}\,. \tag{3.6}$$

This abelian solution is gauge equivalent to the following one:

$$W = \frac{1}{\sqrt{2}\sin\frac{\pi k}{N}} S_N^k, \quad S_N = \begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 \\ 0 & 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 0 & 1 & ... & 0 \\ . & . & . & . & ... & . \\ 0 & 0 & 0 & . & ... & 1 \\ 1 & 0 & 0 & . & ... & 0 \end{pmatrix}, \tag{3.7}$$

where the N-dim shift operator S_N is a unitary matrix with eigenvalues $e^{2\pi in/N}$ with $n = 0, 1, 2, ..., N-1$. An irreducible nonabelian BPS solution may develop from this abelian solution with nonzero $\epsilon_{n,n+k}$. Indeed in the next section, we explore this possibility in detail. The BPS charge J_3 for this critical solution is

$$J_3 = \frac{\mu^3}{27}\frac{N}{2\sin^2\frac{\pi k}{N}}\,. \tag{3.8}$$

For large N, the above J_3 approaches $\frac{\mu^3}{27}\frac{N^3}{2\pi^2k^2}$, which is smaller than the maximal value (2.20) for the ellipsoidal case.

For an abelian solution, we draw a line between any pair of λ_i's satisfying the condition (3.4). As λ_i lie on a complex plane, a graph made of those lines can be decomposed to connected graphs. For each connected graph, there is a potential to develop a nonabelian configuration. Of course there is no guarantee that nonabelian solutions can develop. Any part or whole of a nonabelian BPS configuration will be come such a graph whenever some or all of it becomes abelian.

For the $SU(3)$ case with $\sum_i \lambda_i = 0$, let us consider the case where there points are connected by two lines. Each of segment has length $\sqrt{2}$. They could lie on a straight line or get bent and form a letter V shape. They may form an equi-triangle, or form a sharper tipped V shape, get bent completely to be a single segment where two end points overlap. All these abelian solutions have a potential to be nonabelian. For a straight string and equi-triangle cases, the nonabelian solutions are known. For other bent cases, the nonabelian extensions, if they exist, would be an interesting possibility. Now one can see easily that the central charge (2.16) for these configuration takes the maximal value for straight line. Similarly, we conjecture that the central charge (2.16) of connected graphs of N points take the maximal value for the N points lying on a straight line (3.5).

Now let us change our focus to the BPS configurations built on nonabelian vacua. We start from a nonabelian vacuum where L_a is nontrivial and add a small amount of J_3. A BPS configuration close to the vacuum can be approached by perturbation analysis. Of course there is an ellipsoidal solution (2.17) near each vacuum. We deform the vacuum solution by a small deformation,

$$W = L_+ + \delta W \,. \tag{3.9}$$

We first focus on the (N) vacuum case and expand the matrix δW as the sum of irreducible representations of L_a. For example, the (3) vacuum in the $N = 3$ theory, $SU(3)$ generators belongs to 5-dim and 3-dim representations. The N-dim matrices T^l_m, which belong to the l representation, satisfy the commutation relations,

$$[L_\pm, T^l_m] = \sqrt{l(l+1) - m(m\pm 1)}T^l_{m\pm 1}, \;\; [L_0, T^l_m] = mT^l_m \,, \tag{3.10}$$

which is consistent if $T^{l\dagger}_m = (-1)^m T^l_{-m}$. The normalization is $\mathrm{Tr} T^l_m T^{l'}_n \sim \delta_{ll'}\delta_{m+n,0}$. For example $T^1_1 = L_+/\sqrt{2}$, $T^1_0 = -L_0$, $T^1_{-1} = -L_-/\sqrt{2}$ for the obvious $l = 1$ representation. In this basis, the fluctuated W becomes

$$W = L_+ + C^l_m T^l_m, \; \bar{W} = L_- + (-1)^m \bar{C}^l_{-m} T^l_m \tag{3.11}$$

with complex coefficients C^l_m. We remove the gauge degrees of freedom by diagonalizing $[W, \bar{W}] = 2Z$, which leads to the following relations among the coefficients,

$$(-1)^m \bar{C}^l_{-m+1} A_{m-1} + C^l_{m+1} B_{m+1} = 0, \;\; m \neq 0, \tag{3.12}$$

where $A_m = \sqrt{l(l+1)-m(m+1)}$ and $B_m = \sqrt{l(l+1)-m(m-1)}$. This condition implies

$$C^l_{-l} = \text{arbitrary}, \quad C^l_{-l+1} = 0\,, \tag{3.13}$$

and

$$[W,\bar{W}] = 2L_3 - \sqrt{l(l+1)}(\bar{C}^l_1 + C^l_1)T^l_0\,. \tag{3.14}$$

The left-hand side of the BPS equation (2.13) becomes

$$\begin{aligned} l.h.s. &= 8L_3 - 2(l^2+l+2)\sqrt{l(l+1)}(C^l_1+\bar{C}^l_1)T^l_0 \\ -2\sum_{m\neq 0}&\Big[(-m+2)(-1)^m\bar{C}^l_{-m+1}A_{m-1} + (m+2)C^l_{m+1}B_{m+1}\Big]T^l_m\,. \end{aligned} \tag{3.15}$$

Equating the above expression with four times the expression in Eq. (3.14), we get that for $l \neq 1$ C^l_m vanishes for all m except $m=-l$, and for $l=1$, C^1_1, C^1_{-1} can be arbitrary and C^1_0 vanishes.

Thus from the vacuum (N), the BPS configurations near the vacuum $X_a = L_a$ should be given as

$$W = L_a + \sum_{l\neq 1} C^l_{-l}T^l_{-l} + C^l_1 T^1_1 + C^1_{-1}T^1_{-1}\,, \tag{3.16}$$

where one can sum over all l which appear when the $SU(N)$ generators are split into irreducible representation of irreducible $SU(2)$ algebra. Note that by gauge transformation and $U(1)$ rotation one can make C^1_1 and C^1_{-1} to be real. The above analysis shows that the ellipsoid solution $W = c_1L_+ + c_2L_-$ grows out of the above fluctuation with nonzero C^1_1, C^1_{-1}.

Let us turn our attention to other vacua, for example, the vacuum $(p_1, p_2, ..., p_K)$ with $\sum_k p_k = N$, in which case $L_a = L^1_a \oplus L^2_a \oplus ...$ with each L^k_a being p_k-dim irreducible representation. Our linear fluctuation analysis can be extended directly. The fluctuation is replaced by $C^{l_1l_2..}_{m_1m_2..}T^{l_1l_2..}_{m_1m_2..}$ where (l_k, m_k) indicates the representation of the k-th irreducible part of L^k_a. A coefficient can be nonzero if for all k either $m_k = -l_k$ for or $l_k = 1, m_k = 1$. This is how a BPS solution will develop when a small amount of J_3 charge is added to a nonabelian vacuum.

4. Some Exact BPS Solutions

Let us try to find some (new) exact solutions of the BPS equation (2.13), that is, of

$$\frac{1}{2}[W^2,\bar{W}^2] = (W\bar{W})^2 - (\bar{W}W)^2 - [W,\bar{W}]\,. \tag{4.1}$$

One could, for example, assume two polynomial equations

$$\frac{1}{2}W^2\bar{W}^2 = (W\bar{W})^2 - W\bar{W} + f(W\bar{W},\bar{W}W)\,, \tag{4.2}$$

$$\frac{1}{2}\bar{W}^2W^2 = (\bar{W}W)^2 - \bar{W}W + f(W\bar{W},\bar{W}W)\,, \tag{4.3}$$

where $f(W\bar{W}, \bar{W}W)$ is an arbitrary polynomial of $W\bar{W}$ and $\bar{W}W$. Depending on the function f, the above equations (4.2) and (4.3) could be independent and over-constraining. Our favorite example is

$$f = \alpha \bar{W} W^2 \bar{W} + \beta W \bar{W}^2 W + q \tag{4.4}$$

with real constants α, β, q. Requiring the hermicity of f leads to either $\alpha = \beta$ or $[W\bar{W}, \bar{W}W] = 0$.

The above polynomial equations are a slightly more general non-singular variant of the commutation relation

$$[Z, W] = W - \frac{q}{\bar{W}}, \tag{4.5}$$

that were used in [5, 6] to solve the BPS equation (2.13). By multiplying $\bar{W}$ on the above equation from the right or left, we get Eqs. (4.2) and (4.3) with f in Eq. (4.4) with $\alpha = -1/2$ and $\beta = 0$. Thus we get $[W\bar{W}, \bar{W}W] = 0$ for the BPS configurations to satisfying (4.5). As noted in Ref. [5], the BPS equations (2.13) can be solved partially if there exists an analytic function F such that

$$[Z, W] = W + F(\bar{W})\,. \tag{4.6}$$

Assuming that W is invertible and that there is a $U(1)$ phase rotation symmetry in the above equation, one gets (4.5) in general.

We consider that the $[W\bar{W}, \bar{W}W] =$ case in Eqs. (4.1,4.2, 4.3) is very interesting and may leads to the new type of solutions. But we will not pursue this direction in this work. We will focus here on the simpler case (4.5) which has a $U(1)$ symmetry. When $q = 0$, W, Z satisfy the $SU(2)$ algebra and so the above equation degenerates to the vacuum equation. As it has a $U(1)$ symmetry and $W \neq 0$ for $q \neq 0$, we call the BPS configurations satisfying the above equation as "of toroidal type". Multiplying by $\bar{W}$ and taking the trace, we get

$$qN = \mathrm{Tr}(W\bar{W} - 2Z^2)\,. \tag{4.7}$$

The central charge of any configuration of such ansatz would be

$$J_3 = \frac{\mu^3}{27}\mathrm{Tr}(W\bar{W} - 2Z^2) = \frac{qN\mu^3}{27} \tag{4.8}$$

which should be positive. Thus we restrict to $q > 0$.

We try to solve the above torus-type equation (4.5) with the ansatz

$$W = \begin{pmatrix} 0 & w_1 & 0 & \ldots & 0 \\ 0 & 0 & w_2 & \ldots & 0 \\ . & . & . & . & . \\ 0 & .. & \ldots & 0 & w_{N-1} \\ w_N & 0 & .. & .. & 0 \end{pmatrix}, \tag{4.9}$$

so that $W_{ij} = w_i \delta_{i+1,j}$, mod N. Here we find some additional new solutions and some new generalizations. Both $W\bar{W}$ and $\bar{W}W$ are diagonal and

$$(W\bar{W})_{ii} = |w_i|^2 \equiv r_i, \;\; (\bar{W}W)_{ii} = |w_{i-1}|^2 \equiv r_{i-1} \,. \tag{4.10}$$

Consistency then requires the of alpha and beta to be -1/2, and

$$r_i \left(\frac{r_{i-1} + r_{i+1}}{2} \right) = r_i^2 - r_i + q \,. \tag{4.11}$$

If W is invertible so that all $r_i \neq 0$, we divide the above equation by r_i to get

$$r_i + \frac{q}{r_i} = \frac{r_{i-1} + r_{i+1}}{2} + 1 \,. \tag{4.12}$$

When we sum over i, we get the constraint

$$\frac{1}{N} \sum_i \frac{1}{r_i} = \frac{1}{q} \,, \tag{4.13}$$

which implies that the average of the inverse $1/r_i$ is $1/q$.

The obvious solution would be $r_i = q$ independent of i, which is abelian as shown in Eqs.(3.6) and (3.7) with $\lambda_k = \sqrt{q} e^{i2\pi k/N}$. The analysis in the previous section implies that there may be nonabelian solutions $\{r_1, r_2, ..., r_N\}$ near this constant solution if q is close to the abelian value $q = 1/(2\sin^2(\pi k/N))$.

Let us find some explicit solution by starting with $N = 2$ case. The explicit solution for the equation (4.12) for $N = 2$ case is

$$r_1 = \frac{1 \pm \sqrt{1-2q}}{2}, \;\; r_2 = \frac{1 \mp \sqrt{1-2q}}{2} \,. \tag{4.14}$$

This is the ellipsoidal solution (2.17) with $N = 2$ as we can identify $c_1^2 = r_1, c_2^2 = r_2$. There are two obvious generalization of this $N = 2$ solution. First is somewhat trivial as one finds the solution for all even $N = 2K$ with the periodic condition $r_{k+2} = r_k$ for all $k = 1, 2, .., K$. Another one is the ellipsoidal solution $W = c_1 L_+ + c_2 L_-$ as in Eq. (2.17) where L_a are N-dim representation of $SU(2)$. If we consider an irreducible N-dim representation of $SU(2)$, we know that this generalized BPS configuration is also irreducible. This solution is not new nor does not belong to the toroidal type (4.9).

For $N = 3$, there is no solution with all different r_1, r_2, r_3. The type of solution that appeared in Ref. [6] was

$$r_1 = r_2 = 1 \pm \sqrt{1 - \frac{4q}{3}}, \;\; r_2 = \frac{1}{2}\left(1 \mp \sqrt{1 - \frac{4q}{3}}\right) \,, \tag{4.15}$$

where $0 \leq q \leq 3/4$. As in $N = 2$ case, we have two generalization of this solution. The first one is the periodic extension for all $N = 3K$ with $r_{k+3} = r_k$. Another one

is to rewrite the above solution in terms of $SU(3)$ generators and to generalize the solution to the N-dim presentation of the $SU(3)$ Lie-algebra. The above solution is rewritten as

$$W = c_1 L_+ + c_2 P_- \,, \tag{4.16}$$

where real c_1, c_2 satisfy $c_1^2 + c_2^2 = 1$ with $c_1^2 = r_1/2, c_2^2 = r_3$, and

$$L_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.17}$$

With the definition $L_- = L_+^\dagger$ and $P_+ = P_-^\dagger$, we note that $[P_+, P_-] = -L_3$ with $L_3 = \mathrm{diag}(1, 0, -1)$ and $[L_3/2, P_+] = P_+$. Thus $P_+, P_-, L_3/2$ are another $SU(2)$ generators in $SU(3)$ lie algebra. Thus,

$$Z = (c_1^2 - \frac{c_2^2}{2}) L_3 = \frac{3c_1^2 - 1}{2} L_3 \,. \tag{4.18}$$

The conserved central charge becomes

$$J = \frac{\mu^3}{27} \frac{3c_1^2(1 - c_1^2)}{2} \sum_a \mathrm{Tr}(L_a)^2 \,, \tag{4.19}$$

where we used the isotropy to show $\mathrm{Tr}(L_1 P_1 - L_2 P_2) = \mathrm{Tr}(L_1 P_2 + L_2 P_1) = 0$ and $\mathrm{Tr} L_i^2 = 4 \mathrm{Tr} P_i^2 = \sum_a L_a^2/3$. We see that when $N = 3$ this solution interpolates between two vacua, (3) at $c_1 = 1$ and (2,1) at $c_1 = 0$, via the abelian BPS solution at $c_1 = 1/\sqrt{3}$. Note that $c_1 = 1/\sqrt{3}$, $r_1 = r_2 = r_3 = 2/3$ and $q = 2/3$, and so the solution becomes abelian. Our solution can be regarded as a nonabelian BPS solution growing out of this abelian solution. The maximum value of the central charge for this type of solution appears at $c_1^2 = 1/2$ which is not the abelian case $c_1^2 = 1/3$. Thus one can see that the abelian solution can be developed to the nonabelian ones by either adding or subtracting the central charge. Still the maximum value of the central charge of this type is smaller than that of the abelian limit of the ellipsoidal type (2.17).

Now we generalize L_+ and P_- to an arbitrary N-dim representation of $SU(3)$. For example the symmetric K product of **3**-dim representation of $SU(3)$ would be $N = (K+1)(K+2)/2$ dimensional irreducible representation of $SU(3)$. The adjoint representation is 8-dimensional. For general integer N, the representation would not be irreducible. Even when we have an N-dim irreducible representation of $SU(3)$, we do not have a maximal N-dim representation of $SU(2)$ generator L_a. For $N = 3$ solution, L_+ is for 3-dim representation of $SU(2)$ in $SU(3)$ and P_a is for 2-dim representation of $SU(2)$ in $SU(3)$. The 6-dim irreducible representation of $SU(3)$, for example, belongs to the reducible 5+1-dim representations of L_a and the reducible $3 + 2 + 1$-dim representations of P_a. Thus our BPS solution interpolates two vacua,

(5, 1) at $c_1 = 1$ and (3, 2, 1) at $c_1 = 0$, via an abelian BPS solution at $c_1 = 1/\sqrt{3}$. By going to higher N-dim representation, our fuzzy geometry becomes dense and could goes to continuum once an appropriate scaling is taken. This process could fix the genus uniquely. Of course we expect the present solution has torus topology in the continuum limit.

Now for $N = 4$, there are two types of solutions given in Ref. [6]. There is no solution where all r_i are different. First one is the case where $r_1 = r_2 = 1 + \sqrt{1-q}$ and $r_3 = r_4 = 1 - \sqrt{1-q}$ where $0 \le q \le 1$. This solution can be generalized to $N = 4K$ cases with $r_{a+4} = r_a$. Also similar to $SU(2)$ and $SU(3)$ cases, one can reexpress this solution as

$$W = c_1 L_+ + c_2 P_- \,, \tag{4.20}$$

where real c_1, c_2 satisfy $c_1^2 + c_2^2 = 1$, and

$$L_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 0 & 0 \end{pmatrix}. \tag{4.21}$$

Note that $[L_+, L_-] = [P_+, P_-] = 2L_3$ where $L_3 = \mathrm{diag}(1, 0, -1, 0)$. The commutation relation leads to

$$Z = (2c_1^2 - 1)L_3 \,. \tag{4.22}$$

The conserved central charge becomes

$$J_3 = \frac{\mu^3}{27}\frac{8}{3}c_1^2(1 - c_1^2)\mathrm{Tr}L_a^2 \,. \tag{4.23}$$

This solution interpolates the vacuum (3,1) at $c_1 = 0, 1$ to itself via an abelian BPS solution at $c_1 = 1/\sqrt{2}$. Again, we can consider N-dim representation of $SU(4)$ and generalize the above solution to such a space. For example, the adjoint representation would be 15-dimensional. The symmetric product of two 4-dim representation would be 10-dim, and the anti-symmetric product of two 4-dim representation would be 6-dim.

The second type of solution with $N = 4$ is

$$\begin{aligned} & r_1 = r_3 = \frac{1}{2}(3 \pm \sqrt{9 - 8q}), \; r_2 = \frac{1}{2}(r_1 + 1 \pm \sqrt{(r_1 + 1)2 - 4q}), \\ & r_4 = \frac{1}{2}(r_1 + 1 \mp \sqrt{(r_1 + 1)2 - 4q}) \,. \end{aligned} \tag{4.24}$$

Again this solution can be generalized to $N = 4K$ with $r_{a+4} = r_a$. We reexpress the above solution as

$$W = c_1 L_+ + c_2 M_+ + c_3 P_- \,, \tag{4.25}$$

where

$$L_+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; M_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; P_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.26}$$

Here the coefficients c_i are given by the relations, $c_1^2 = r_1, c_2^2 = r_2, c_3^2 = r_2$, and satisfy the constraints $c_1^2+1 = c_2^2+c_3^2$ and $c_1^2(3-c_1^2) = 2c_2^2c_3^2$. The range of c_1 is $0 \le c_1^2 \le 1/3$ or $1 \le c_1^2 \le 3$. The commtation of $[W, \bar{W}]$ leads to

$$Z = c_1^2 L_3 + c_2^2 M_3 - c_3^2 P_3 \,, \tag{4.27}$$

where

$$L_3 = \frac{1}{2}\mathrm{diag}(1,-1,1,-1), \;\; M_3 = \frac{1}{2}\mathrm{diag}(0,1,-1,0), \;\; P_3 = \frac{1}{2}\mathrm{diag}(1,0,0,-1)\,. \tag{4.28}$$

The central charge becomes

$$J = \frac{\mu^3}{27}\frac{4}{3}c_1^2(3-c_1^2)\mathrm{Tr}M_a^2 \,, \tag{4.29}$$

where we have used $L_3 = -M_3 + P_3$ and $\mathrm{Tr}L_a^2 = 2\mathrm{Tr}M_a^2 = 2\mathrm{Tr}P_a^2$. When $c_1^2 = 0$, $(c_2^2, c_3^2) = (0,1)$ or $(1,0)$, the solution becomes the $(2,1,1)$ vacuum. When $c_1^2 = 3$, $(c_2^2, c_3^2) = (0,4)$, or $(4,0)$ the solution becomes the (4) vacuum. When $c_1^2 = 1/3$, $c_2^2 = c_3^2 = 2/3$, it is a nonabelian BPS solution obtained from the N=2 case with the generalization $r_{i+2} = r_i$. When $c_1^2 = 1$, $c_2^2 = c_3^2 = 1$ and so that $Z = 0$ and so the solution becomes the abelian BPS solution.

For N=5, one can consider type of solution $r_1 = r_4, r_2 = r_3, r_5$ type of solutions which is invariant under the reflection of a pentagon. One gets a 7th order equation which is hard to solve. However, one can in principle generalize this solution to the operator equations,

$$W = c_1 L_+ + c_2 M_+ + c_3 P_- \,, \tag{4.30}$$

where

$$L_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \;\; M_+ = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \;\; P_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.31}$$

The coefficients c_i satisfy the relations, $(c_2^2+1)c_3^2 - c_3^4 = (2+c_2^2)c_1^2 - 2c_1^4$ and $2(c_2^2 + c_1^2)c_3^2 - 2c_3^4 = (2 + 2c_1^2 + c_3^2)c_2^2 - 2c_2^4$. These equations are easer to solve, say for a given value of c_3.

The equation for r_i becomes more and more involved as N increases. We find a new exact solution with $N = 6$, which does not appear in Ref. [6] , such that

$$\begin{aligned} r_3 &= r_6 = 2 \pm \sqrt{4-2q}\,, \\ r_1 &= r_2 = (1+r_3/2) \pm \sqrt{(1+r_3/2)2-2q}\,, \\ r_4 &= r_5 = (1+r_3/2) \mp \sqrt{(1+r_3/2)2-2q}\,. \end{aligned} \tag{4.32}$$

In terms of matrices, the solution can be written as

$$W = c_1 L_+ + c_2 M_+ + c_3 P_+ \ ,, \tag{4.33}$$

where

$$L_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.34}$$

The coefficients are given as $2c_1^2 = r_1 = r_2$, $2c_2^2 = r_4 = r_5$ and $c_3^2 = r_3 = r_6$, and so satisfy the constraints, $c_1^2 + c_2^2 = 1 + c_3^2/2$, and $4c_1^2c_2^2 = c_3^2(4 - c_3^2)$. The range is given by $0 \le c_3^2 \le 2/5$ or $2 \le c_3^2 \le 4$. The commutation relation $[W, \bar{W}]$ leads to

$$Z = c_1^2 L_3 + c_2^2 M_3 + c_3^2 P_3\,, \tag{4.35}$$

where

$$\begin{aligned} L_3 &= \mathrm{diag}(1, 0, -1, 0, 0, 0)\,, \\ M_3 &= \mathrm{diag}(0, 0, 0, 1, 0, -1)\,, \\ P_3 &= \frac{1}{2}\mathrm{diag}(-1, 0, 1, -1, 0, 1)\,. \end{aligned} \tag{4.36}$$

Note that $P_3 = -(L_3 + M_3)/2$. The central charge becomes

$$J_3 = \frac{\mu^3}{27}\frac{2}{3}\Big(\big(c_1^2 + \frac{c_3^2}{4} - (c_1^2 - \frac{c_3^2}{2})^2\big)\mathrm{Tr}L_a^2 + \big(c_2^2 + \frac{c_3^2}{4} - (c_2^2 - \frac{c_3^2}{2})^2\big)\mathrm{Tr}M_a^2\Big)\,. \tag{4.37}$$

This solution interpolates the vacuum (3,1,1,1) at $c_1^2 = 1, c_2^2 = 0, c_3^2 = 0$ or $c_1^2 = 0, c_2^2 = 1, c_3^2 = 0$ and the vacuum (5,1) at $c_1^2 = 3, c_2^2 = 0, c_3^2 = 4$ or $c_1^2 = 0, c_2^2 = 3, c_3^2 = 4$, via the abelian solution at $c_1^2 = c_2^2 = 1, c_3^2 = 2$. Again this solutions can be generalized to arbitrary dimension by considering N-dim representation of the $SU(6)$ generators

5. Conclusion

In this work, we investigated the 1/2 BPS configurations with $SO(3)$ angular momentum in the BMN matrix theory. From the abelian BPS configurations, we have seen how nonabelian field configurations can emerge and made a conjecture on the exact value of the maximum angular momentum J_3 as a function of N for irreducible nonabelian configurations. From the fluctuation analysis of the BPS configurations around the nonabelian vacuum we learned how nontrivial solutions can emerge from a given fuzzy sphere. Finally we found some exact 1/2 BPS configurations which are new because of the further interpretation of the already known ones or some new type of solutions.

The general solutions of the 1/2 BPS equations have been found in the continuum limit in Ref. [6]. The BPS solutions can be characterized by the Riemann surface with arbitrary number of genus and spikes. For finite N, it is more difficult to assign the genus to the fuzzy object. For an irreducible nonabelian BPS configuration in a given N, one can take a unique large N and continuing limit by taking the higher dimensional irreducible representation of the same configuration. This may let us assign a unique genus number to a given irreducible fuzzy object. Our exact BPS solutions are fuzzy tori type of genus 1. A variation of our approach may leads to more involved fuzzy Riemann surfaces. This remains to be seen.

The 1/2 BPS deformations of the maximally supersymmetric AdS geometries in M theory have been studied extensively [16]. Our 1/2 BPS configurations in the BMN matrix would correspond to the 1/4 BPS deformation of the AdS geometries. A direct supersymmetric geometry for BMN matrix theory has been studied [19]. Other classes of 1/4 BPS deformations of the AdS geometry in M theory have been also explored recently [17, 18]. It would be interesting to find the geometric counterpart for the BPS fuzzy Riemann surfaces studied here.

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